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# On the cohomology of the noncritical $W$ -string

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## Abstract

We investigate the cohomology structure of a general noncritical  $W_N$  string. We do this by introducing a new basis in the Hilbert space in which the BRST operator splits into a “nested” sum of nilpotent BRST operators. We give explicit details for the case  $N = 3$ . In that case the BRST operator  $Q$  can be written as the sum of two, mutually anticommuting, nilpotent BRST operators:  $Q = Q_0 + Q_1$ . We argue that if one chooses for the Liouville sector a  $(p, q)$   $W_3$  minimal model then the cohomology of the  $Q_1$  operator is closely related to a  $(p, q)$  Virasoro minimal model. In particular, the special case of a  $(4, 3)$  unitary  $W_3$  minimal model with central charge  $c = 0$  leads to a  $c = 1/2$  Ising model in the  $Q_1$  cohomology. Despite all this, noncritical  $W_3$  strings are not identical to noncritical Virasoro strings.

## 1. Introduction

Noncritical strings are strings in which the two-dimensional gravitational fields do not decouple after quantization but instead develop an induced kinetic term. The string coordinates are called “matter” fields while the non-decoupled gravity fields are represented by a set of so-called “Liouville” fields. The matter and Liouville fields together form a realization of the Virasoro algebra. The spectrum of the noncritical string in less than or precisely one dimension has been calculated via a BRST analysis in refs. [1–3].

Unfortunately, if the number of dimensions in which the noncritical string propagates is larger than one, tachyons appear in the spectrum, and the theory

is ill-defined. It is believed that it is possible to make sense out of noncritical strings in more than one dimension, provided the underlying Virasoro algebra is extended to a nonlinear so-called  $W$ -algebra [4]. In the last few years much research has been devoted to investigating the structure of these  $W$ -symmetries (for a recent review, see e.g. ref. [5]). Noncritical string theories with an underlying  $W$ -symmetry are referred to as noncritical  $W$ -strings.

One expects that in a noncritical  $W$ -string the matter and Liouville fields (representing the “ $W$ -gravity” sector) separately form a realization of the underlying nonlinear  $W$ -algebra. Thus we are faced with an immediate problem: due to the nonlinear nature of the  $W$ -algebra it is clear that the matter and Liouville fields *together* do not realize the same  $W$ -algebra. This would seem to make the construction of a nilpotent BRST operator problematic. A way out of this apparent obstacle was provided by the work of ref. [6] where it was shown that nevertheless a nilpotent BRST operator for the matter + Liouville system could be constructed. This is made possible by the fact that at the classical level the sums of the generators of the  $W$ -algebras in the matter and Liouville sectors still form a closed Poisson bracket algebra, albeit with field-dependent structure functions [7]. For definiteness, we will call this algebra the “modified”  $\tilde{w}$ -algebra<sup>1</sup>. Alternatively, the existence of this BRST operator can be understood from the covariant action for  $W$ -gravity coupled to matter [8].

Having a nilpotent BRST operator at one’s disposal one can proceed with a calculation of its cohomology and thus the spectrum of the noncritical  $W$ -string. Some results in this direction have been obtained in refs. [6,9–12].

Recently, it has been shown that the BRST operator of an “unmodified”  $W_3$  algebra<sup>2</sup> can be decomposed as the sum of two separate, mutually anticommuting, BRST operators by performing a particular canonical transformation [14]. This is related to the fact that after the canonical transformation the spin-3 generators of the *classical*  $w_3$  algebra form a closed Poisson bracket algebra with field-dependent structure constants [15]. It turns out that quite generally the BRST operator corresponding to a  $W_N$  algebra can be written as a “nested” sum of nilpotent BRST operators and the same applies to the BRST operator of the noncritical  $W$ -string [16].

One of the advantages of decomposing a BRST operator into a “nested” sum of nilpotent BRST operators, is that one can now construct the cohomology by an iterative procedure. One first calculates the cohomology of one of the BRST operators occurring in the sum. In practice it is easiest to start with the BRST operator corresponding to the highest-spin generator (see ref. [16] for more details). One then includes the next to highest-spin generator and the coho-

<sup>1</sup> We denote the quantum extension of a classical  $w$ -algebra, provided it exists, by a  $W$ -algebra.

<sup>2</sup> This case is often referred to as the “critical”  $W$ -string. The possibility of constructing new critical string theories by exploiting the  $W$ -symmetries was first suggested in ref. [13]. Note that the “critical”  $W$ -string can be obtained from the “noncritical”  $W$ -string by setting the Liouville fields equal to zero.

mology of the combined system can be obtained from the one corresponding to the highest spin generator by a so-called “tic-tac-toe” procedure [17].

In refs. [18,19] a relationship has been suggested, on the basis of comparing values of central charges, between the spectra of “critical”  $W_N$  strings and Virasoro minimal models. In the case of the  $W_3$  string this relation has been made more explicit in refs. [14,20–23]. In particular, it was shown that a general physical state of the critical  $W_3$  string contains a factor corresponding to a  $c = 1/2$  Ising model. In terms of the new basis discussed above, it means that the cohomology of the BRST-operator of the highest-spin symmetry corresponds to a  $c = 1/2$  Ising model.

Using similar arguments as in ref. [18,19], it can be shown that the corresponding situation for the noncritical  $W_N$  string is as follows [16]. Given a realization of the matter and Liouville sectors in terms of  $N - 1$  free scalars, one can always perform a canonical transformation in the matter sector<sup>1</sup> such that the “modified”  $\tilde{w}_N$ -algebra manifestly has a “nested” set of subalgebras

$$v_N^N \subset v_N^{N-1} \subset \dots \subset v_N^2 \equiv \tilde{w}_N, \quad (1)$$

where the subalgebra  $v_N^n$  consists of generators of spin  $s = \{n, n + 1, \dots, N\}$ , respectively. Each generator of spin  $s = n$  depends on  $N - n + 1$  of the  $N - 1$  matter scalars and all the  $N - 1$  Liouville scalars. In the new basis the BRST charge  $Q_N$  of the  $\tilde{w}_N$  algebra has the following nested structure:

$$Q_N^N \subset Q_N^{N-1} \subset \dots \subset Q_N^2 \equiv Q_N, \quad (2)$$

where  $Q_N^n$  is the BRST charge corresponding to the subalgebra  $v_N^n$ . The BRST charge  $Q_N^n$  depends on  $N - n + 1$  matter scalars, all the  $N - 1$  Liouville fields and the ghost and anti-ghost fields of the spin  $n, \dots, N$  symmetries. The inclusion symbols in (2) indicate that the BRST charge  $Q_N^n$  can be obtained from the BRST charge  $Q_N^{n-1}$  by setting in the expression for  $Q_N^{n-1}$  the ghosts and antighosts corresponding to the spin- $(n - 1)$  symmetries equal to zero. In all explicit examples considered thus far the nested structure of the BRST charges (2) survives quantization, where the BRST charges become BRST operators. To distinguish between BRST operators and BRST charges, we will write the operators with boldface<sup>2</sup>. We note that the case  $N = 3$  is special. In that case the BRST operator decomposes as the sum of two, mutually anticommuting, nilpotent BRST operators as follows:

$$Q_3 = Q_0 + Q_1, \quad (3)$$

<sup>1</sup> The discussion below can be repeated if one performs the canonical transformation in the Liouville sector.

<sup>2</sup> The explicit form of the  $Q_N^N$  operator for  $N = 4, 5, 6$  (with the Liouville fields set equal to zero) is given in ref. [24] while the explicit expression for the  $Q_4^3$  operator (without zero Liouville fields) can be found in ref. [16]. In the usual Miura basis [25,18], the BRST operator of the  $W_4$  algebra was given in refs. [26,27].

where  $Q_1 = Q_3^3$  and  $Q_0$  is the nilpotent BRST operator corresponding to the Virasoro subalgebra.

By comparing the values of central charges, it can now be argued that, if one restricts the Liouville sector of a noncritical  $W_N$  string to a  $(p, q)$   $W_N$  minimal model, then the cohomology of the  $Q_N^n$  BRST operator is closely related to the  $(p, q)$  minimal model of the  $W_{n-1}$  algebra.

More precisely, the counting of the central charges is as follows [16]. Both for the matter and the Liouville sectors we realize the  $W_N$  algebra by  $N - 1$  scalars  $\phi_k$  and  $\sigma_k$ , respectively. The background charges of these fields are fixed by the Miura transformation [25,18]. Consider now the  $Q_N^n$  operator. It depends on  $N - n + 1$  matter scalars, the Liouville fields and the ghosts and antighosts for the spin  $n, \dots, N$  symmetries. In general, the central charge contribution of the spin  $n, \dots, N$  ghosts and antighosts for a given value of  $n$  is given by

$$c_{\text{gh}}^n = -2 \sum_{k=n}^N (6k^2 - 6k + 1). \quad (4)$$

In particular, the contribution of *all* ghosts and antighosts is given by

$$c_{\text{gh}} \equiv c_{\text{gh}}^2 = -2(N - 1)(2N^2 + 2N + 1). \quad (5)$$

The central charge contributions of the Liouville and matter sectors are given by

$$\begin{aligned} c_L &= (N - 1) \left\{ 1 - 4(Q^2 - N(N - 1)) \frac{N + 1}{N - 1} \right\}, \\ c_m &= (N - 1) \left\{ 1 + 4Q^2 \frac{N + 1}{N - 1} \right\}, \end{aligned} \quad (6)$$

where  $Q$  is a free parameter which can be identified with the background charge of the matter field  $\phi_{N-1}$ . Note that we have

$$c_m + c_L + c_{\text{gh}} = 0, \quad (7)$$

as is required to allow for a nilpotent BRST operator. We now choose for  $Q$  one of the following values:

$$Q_{\text{min}}^2 = N(N - 1) \frac{(p + q)^2}{4pq}, \quad (8)$$

where  $p$  and  $q$  are non-negative integers which are relatively prime. The central charge contribution of the Liouville sector is then given by

$$c_L^{(p,q)} = (N - 1) \left\{ 1 - \frac{(p - q)^2}{pq} N(N + 1) \right\}, \quad (9)$$

which corresponds to the  $(p, q)$  minimal model of the  $W_N$  algebra. Similarly, the central charge contribution of the matter sector is given by

$$c_m^{(p,q)} = (N-1) \left\{ 1 + \frac{(p+q)^2}{pq} N(N+1) \right\}, \quad (10)$$

which corresponds to the  $(p, -q)$  minimal model of the  $W_N$  algebra. Finally, for fixed  $n$ , the central charge contribution of the  $N - n + 1$  matter scalars is given by

$$c_m^n = \sum_{k=n-1}^{N-1} (1 + 12(\alpha_k)^2), \quad (11)$$

where  $\alpha_k$  is the background charge of the matter scalar  $\phi_k$ :

$$\alpha_k = Q_{\min} \sqrt{\frac{k(k+1)}{N(N-1)}}. \quad (12)$$

Denoting all the different contributions of the fields occurring in the  $\mathbf{Q}_N^n$  operator by  $c_N^{n;(p,q)}$  we find for a given choice of  $p, q$  and  $n$

$$\begin{aligned} c_N^{n;(p,q)} &= c_L^{(p,q)} + c_m^n + c_{\text{gh}}^n \\ &= (n-2) \left\{ (2n-1)^2 - n(n-1) \frac{(p+q)^2}{pq} \right\}, \end{aligned} \quad (13)$$

which corresponds to the  $(p, q)$  minimal model of the  $W_{n-1}$  algebra. This concludes our counting argument.

An interesting special case arises if we choose the Liouville sector to correspond to the  $(N+1, N)$   $W_N$  minimal model with  $c_L^{(N+1,N)} = 0$  [15]. In that case the Liouville fields effectively decouple from the theory and we end up with a “critical”  $W_N$  string. The central charge corresponding to the  $\mathbf{Q}_N^n$  operator is in that case given by

$$c_N^{n;(N+1,N)} = (n-2) \left\{ 1 - \frac{n(n-1)}{N(N+1)} \right\}, \quad (14)$$

which is the  $(N+1, N)$  minimal model of the  $W_{n-1}$  algebra<sup>1</sup>. For instance, for  $n = N = 3$  we find the  $c_3^{3;(4,3)} = 1/2$  Ising model.

Another interesting case occurs if we choose the Liouville sector to correspond to the  $(n, n-1)$  minimal model of the  $W_N$  algebra with central charge ( $2 \leq n \leq N$ )

$$c_L^{(n,n-1)} = (N-1) \left\{ 1 - \frac{N(N+1)}{n(n-1)} \right\}. \quad (15)$$

<sup>1</sup> This relation between critical  $W$ -strings and minimal models has been suggested in refs. [28,23].

In that case we find that  $c_N^{n;(n,n-1)} = 0$ . This means that all fields occurring in the  $Q_N^n$  cohomology decouple and we effectively end up with a “critical”  $W_{n-1}$  string theory. An interesting additional feature of the Liouville central charge given in (15) is that for this value the total central charge contribution of the  $n - 2$  Liouville scalars  $\sigma_1, \dots, \sigma_{n-2}$  equals zero. For instance, if  $n = N$  then  $c_L^{(N,N-1)} = -2$  and the central charge contribution of the  $N - 2$  Liouville scalars  $\sigma_1, \dots, \sigma_{N-2}$  equals zero. In the particular case  $N = 3$  [7], we have two Liouville scalars  $\sigma_1$  and  $\sigma_2$ . The central charge contribution of  $\sigma_1$  is zero and the whole  $W_3$  algebra can in fact be realized in terms of the *single* scalar  $\sigma_2$  [29].

From a group-theoretic point of view, the picture is the following. The matter fields can be seen as one scalar field  $\varphi$  with values in the Cartan subalgebra of  $\mathfrak{sl}_N$ . The energy momentum tensor of these fields can be written as

$$T = -\frac{1}{2} \partial \varphi \cdot \partial \varphi + B \rho_N \cdot \partial^2 \varphi, \quad (16)$$

with central charge  $c = (N - 1) + 12 B^2 \rho_N \cdot \rho_N$ . Here,  $\rho_N$  is half the sum of the positive roots of  $\mathfrak{sl}_N$ . If the Liouville sector is a  $(p, q)$  minimal model, then one finds, on using (10) and  $\rho_N \cdot \rho_N = (N^3 - N)/12$ , that  $B = \sqrt{p/q} + \sqrt{q/p}$ <sup>1</sup>. To get the  $Q_N^n$  operator, we have to decouple  $n - 2$  matter fields, which can be done as follows. Take an embedding of  $\mathfrak{sl}_{n-1}$  into  $\mathfrak{sl}_N$ , for instance map the  $n - 2$  simple roots of  $\mathfrak{sl}_{n-1}$  to the first  $n - 2$  simple roots of  $\mathfrak{sl}_N$ . Then, decompose the field  $\varphi$  as  $\varphi_n + \varphi^\perp$ , where  $\varphi_n$  is the orthogonal projection of  $\varphi$  onto the Cartan subalgebra of  $\mathfrak{sl}_{n-1}$ . The  $n - 2$  matter fields we want to decouple are precisely given by the components of  $\varphi_n$ . To find the energy momentum tensor for  $\varphi_n$ , we have to decompose  $\rho_N$  into a piece with values in the  $\mathfrak{sl}_{n-1}$  Cartan subalgebra and a piece perpendicular to that. But this latter piece is precisely  $\rho_N - \rho_{n-1}$ , because both  $\rho_N$  and  $\rho_{n-1}$  have inner product one with the simple roots  $\alpha_1, \dots, \alpha_{n-2}$ , so that their difference has inner product zero with respect to these simple roots and is therefore perpendicular to the  $\mathfrak{sl}_{n-1}$  Cartan subalgebra. This demonstrates that the energy momentum tensor for  $\varphi_n$  is identical to (16), but with  $\rho_N$  replaced by  $\rho_{n-1}$ . The value of  $B$  remains the same, so that the central charge of the fields  $\varphi_n$  is equal to central charge of the matter fields in a theory of noncritical  $W_{n-1}$  gravity, where the Liouville fields form a  $(p, q)$  minimal model for the  $W_{n-1}$  algebra. This is the same statement as we made previously, and we see that the fields that enter the  $Q_N^n$  operator are those orthogonal to an  $\mathfrak{sl}_{n-1}$  subalgebra of  $\mathfrak{sl}_N$ . Later, we will use this observation to try to improve our understanding of the results of the computation of the  $Q_3^3 \equiv Q_1$  cohomology.

Since the BRST operator for the noncritical  $W_N$  string can be derived via hamiltonian reduction from the superalgebra  $\mathfrak{sl}(N|N - 1)$ , it is an interesting

<sup>1</sup> The parameter  $B$  is related to  $Q_{\min}$  (8) by  $Q_{\min} = \frac{1}{2} \sqrt{N(N - 1)} B$ .

idea to try derive the operators  $Q_N^n$  from hamiltonian reduction as well, employing the group-theoretical insight obtained above.

In the above discussion, all relations between noncritical  $W$ -strings and minimal models have been based upon comparing values of central charges. It is the purpose of this paper to confirm the relations suggested by the above counting arguments for  $N = 3$  by considering the weights of the primary fields occurring in the cohomology. We will verify that these weights indeed correspond to the relevant minimal model. Note that for  $N = 3$  the BRST operator decomposes according to (3). Our main result can be stated as follows. Starting from a  $(p, q)$   $W_3$  minimal model in the Liouville sector, we find that the primary fields occurring in the cohomology of the  $Q_1$  operator exactly correspond to those of the  $(p, q)$  minimal model of the Virasoro algebra. More precisely, all but one of the primary fields occur as tachyonic states at level 0 while the remaining one occurs at level one. We suggest that all other solutions of the cohomology can be obtained by the action of so-called picture-changing and screening operators on this basis set of primary fields.

The organization of the paper is as follows. In section 2 we introduce some basic notions of the noncritical  $W_3$  string. In section 3 we consider the primary fields occurring in the  $Q_1$  cohomology and show that for the choice (8) of the background charge  $Q$  indeed they correspond to  $(p, q)$  minimal models of the Virasoro algebra. In section 4 we consider the screening operators and the picture-changing operators of the  $Q_1$  operator. We will argue that the complete cohomology of  $Q_1$  is obtained by acting on the basic primary fields with strings of screening and picture-changing operators<sup>1</sup>. Next, in section 5 we illustrate our results by working out examples for specific values of  $p$  and  $q$ . Finally, in section 6 we will discuss the cohomology of the  $Q_1$  operator for a  $W_3$  minimal model, the  $Q_0 + Q_1$  cohomology and generalizations to other  $W$ -algebras. In particular we discuss in what way noncritical Virasoro strings are embedded in noncritical  $W_3$  strings. Our results indicate that the  $W_3$   $(p, q)$  minimal model coupled to  $W_3$  gravity is not the same theory as the Virasoro  $(p, q)$  minimal model coupled to ordinary gravity, although it is contained in the noncritical  $W_3$  string. The Virasoro minimal model is recovered by taking a modified noncritical  $W_3$  string, that contains an additional set of screening operators. It is interesting to compare this with the work of Berkovits and Vafa [30], who demonstrated how special  $N = i$  strings reduce to arbitrary  $N = j$  strings,  $j < i \leq 2$ . It is tempting to conjecture that strings based on certain chiral algebras can always somehow be embedded in strings based on larger chiral algebras. For example, it was shown recently [31] that the ordinary bosonic string can be realized as a specific critical  $W_3$  string. This realization differs from ours, since we realize certain bosonic strings as subsectors of non-critical  $W_3$  strings. It would be interesting to see if there is a deeper relation between the two realizations.

<sup>1</sup> This concerns the states with continuous momenta. The situation of the discrete states is more complicated.



Table 1

The fields of the noncritical  $W_3$  string. The scalar fields of the matter sector ( $\phi_1, \phi_2$ ) and of the Liouville sector ( $\sigma_1, \sigma_2$ ) are given with their background charge, and their contribution to the central charge. The fields ( $c, b$ ) are the spin-2, ( $\gamma, \beta$ ) the spin-3 ghosts.

Field	Background charge	Central charge
$\phi_1$	$\frac{1}{3}\sqrt{3}Q$	$1 + 4Q^2$
$\phi_2$	$Q$	$1 + 12Q^2$
$\sigma_1$	$\frac{1}{3}\sqrt{3(6 - Q^2)}$	$25 - 4Q^2$
$\sigma_2$	$\sqrt{6 - Q^2}$	$73 - 12Q^2$
$c, b$		$-26$
$\gamma, \beta$		$-74$

## 2. The noncritical $W_3$ string

In this section we will illustrate the general structure of the quantum BRST operators  $Q_N^n$  for the noncritical  $W_N$  string with the  $N = 3$  case. We will employ a simple realisation of the noncritical  $W_3$  string in terms of two scalars in the matter sector, as well as two scalars for the Liouville or  $W$ -gravity sector, two scalars being the minimum number for a realisation of the  $W_3$  algebra with arbitrary central charge.

In Table 1 we present these fields with the corresponding background and central charges. These charges are limited by the requirement that the total central charge equals 100, thus cancelling the contribution from the spin-2 and spin-3 ghosts, and by the fact that the matter and Liouville sectors separately realize a  $W_3$  algebra. This fixes the relative coefficient between the background charges in these two-scalar realizations [32].

In terms of roots of  $\mathfrak{sl}_3$ ,  $\phi_1$  corresponds to  $\alpha_1$  and  $\phi_2$  to  $2\alpha_2 + \alpha_1$ .

In ref. [15] the BRST operator of the noncritical  $W_3$  string in the new basis, discussed in the introduction, was calculated. It is given by  $Q = \oint j$ , with  $j = j_0 + j_1$  given by

$$j_0 = c \{ T_M + T_L + T_{(\gamma, \beta)} + \frac{1}{2} T_{(c, b)} \}, \quad (17)$$

$$\begin{aligned}
 j_1 = \gamma \bigg[ & \frac{i}{3\sqrt{6}} \{ 4(\partial\phi_2)^3 - 12Q\partial\phi_2\partial^2\phi_2 + (-15 + 4Q^2)\partial^3\phi_2 \} \\
 & + i \{ W_L - \frac{2}{\sqrt{6}}\partial\phi_2 T_L + \frac{1}{\sqrt{6}}Q\partial T_L \} \\
 & - i\sqrt{6} \{ \partial\phi_2\partial\gamma\beta + \frac{1}{3}Q\partial\beta\partial\gamma \} \bigg]. \quad (18)
 \end{aligned}$$

The main advantage of the chosen basis is that  $Q_0 = \oint j_0$  and  $Q_1 = \oint j_1$  are separately nilpotent, and therefore anticommute.

Note that the scalar  $\phi_1$ , and the spin-2 ghosts are absent from  $j_1$ , and that the Liouville fields do not occur explicitly in (18), but only in

$$T_L = T_{\sigma_1} + T_{\sigma_2} \\ = -\frac{1}{2}(\partial\sigma_1)^2 + \frac{1}{3}\sqrt{3(6-Q^2)}\partial^2\sigma_1 - \frac{1}{2}(\partial\sigma_2)^2 + \sqrt{6-Q^2}\partial^2\sigma_2, \quad (19)$$

$$W_L = \frac{1}{18}i\sqrt{6}\left\{(\partial\sigma_2)^3 - 3\sqrt{6-Q^2}\partial\sigma_2\partial^2\sigma_2 \right. \\ \left. + (6-Q^2)\partial^3\sigma_2 + 6\partial\sigma_2T_{\sigma_1} - 3\sqrt{6-Q^2}\partial T_{\sigma_1}\right\}. \quad (20)$$

$T_L$  and  $W_L$  satisfy a  $W_3$  algebra. In  $j_0$  we find, besides  $T_L$ , the energy-momentum tensors of the matter and ghost sectors:

$$T_M = T_{\phi_1} + T_{\phi_2} \\ = -\frac{1}{2}(\partial\phi_1)^2 + \frac{1}{3}\sqrt{3}Q\partial^2\phi_1 - \frac{1}{2}(\partial\phi_2)^2 + Q\partial^2\phi_2, \quad (21)$$

$$T_{(c,b)} = -2b\partial c - (\partial b)c, \quad (22)$$

$$T_{(\gamma,\beta)} = -3\beta\partial\gamma - 2(\partial\beta)\gamma. \quad (23)$$

For the convenience of the reader we will give some of the formulae of the introduction specifically for  $N = 3$ . The total Liouville central charge  $c_L$ , and  $c_3^3$ , the contribution to the central charge of the fields that play a role in  $\mathbf{Q}_1$  ( $\phi_2$ , the spin-3 ghosts, and the Liouville scalars) are given by

$$c_L = 98 - 16Q^2, \quad (24)$$

$$c_3^3 = 25 - 4Q^2. \quad (25)$$

If we choose  $Q$  equal to

$$Q_{\min} = \frac{3(q+p)}{\sqrt{6pq}}, \quad \sqrt{6-Q^2_{\min}} = \frac{3i(q-p)}{\sqrt{6pq}}, \quad (26)$$

we find the following values for  $c_L$  and  $c_3^3$ :

$$c_L^{(p,q)} = 2\left(1 - \frac{12(p-q)^2}{p\bar{q}}\right), \quad (27)$$

$$c_3^{3;(p,q)} = 1 - \frac{6(p-q)^2}{pq}. \quad (28)$$

The values  $c_L^{(p,q)}$  and  $c_3^{3;(p,q)}$  correspond to the central charges of the  $(p,q)$   $W_3$  minimal model, and the  $(p,q)$  Virasoro minimal model, respectively. This relationship between  $W_3$  and Virasoro minimal models will be further elucidated in the next two sections, where we will consider the cohomology of  $\mathbf{Q}_1$ .

With the above formulae for  $N = 3$ , it is a simple matter to verify the relations between central charges in the special cases considered in the introduction. After treating the  $\mathbf{Q}_1$  cohomology in the next two sections, we will come back to these examples in section 5.

### 3. The $Q_1$ cohomology

In this section we will consider the cohomology of the operator  $Q_1$ , acting on Fock spaces of  $\phi_2, \sigma_1, \sigma_2$  and of the ghosts  $\beta, \gamma$ . The momenta of  $\sigma_1$  and  $\sigma_2$  will be chosen to be equal to those appearing in the Felder resolution of a  $W_3$  minimal model. In section 6 we explain the implications of these results for the computation of the  $Q_1$  cohomology when the Fock space of  $\sigma_1, \sigma_2$  is replaced by the Hilbert space of a  $W_3$  minimal model. In particular, for the values of the background charge  $Q$  corresponding to a  $W_3$  minimal model in the Liouville sector, and the momenta of  $\sigma_1, \sigma_2$  equal to those in the corresponding Felder resolution, we want to identify the primary operators of the Virasoro minimal model with central charge (28).

For this it turns out to be sufficient to consider the states at level 0 and level 1. At level 0 the states of lowest ghost number are of the form<sup>1</sup>

$$V_0(p_2, s_1, s_2) = (\partial\gamma)\gamma e^{ip_2\phi_2 + is_1\sigma_1 + is_2\sigma_2}. \quad (29)$$

The condition  $Q_1 V_0(p_2, s_1, s_2) = 0$  determines the momenta of the three fields. The resulting cubic equation factorizes, and we obtain the following three solutions:

$$(A_0) \quad p_2 = is_2 - iQ - \sqrt{6 - Q^2}, \quad (30)$$

$$(B_0) \quad p_2 = +\frac{1}{2}i\sqrt{3}s_1 - \frac{1}{2}is_2 - iQ, \quad (31)$$

$$(C_0) \quad p_2 = -\frac{1}{2}i\sqrt{3}s_1 - \frac{1}{2}is_2 - iQ + \sqrt{6 - Q^2}, \quad (32)$$

where  $s_1$  and  $s_2$  are arbitrary.

Now, if we choose the parameter  $Q$  equal to (26), the central charge of the Liouville sector corresponds to that of a  $(p, q)$   $W_3$  minimal model. By choosing  $s_1$  and  $s_2$  appropriately, we restrict these momenta to those of this minimal model, and in that case we expect that the states determined by (30)–(32) should correspond to the primary states of the  $(p, q)$  minimal Virasoro model. The allowed values of  $s_1$  and  $s_2$  are (see, e.g., ref. [5])

$$s_1 = -\frac{1}{\sqrt{2pq}}(qr_2 - pt_2), \quad (33)$$

$$s_2 = -\frac{1}{\sqrt{6pq}}(2(qr_1 - pt_1) + (qr_2 - pt_2)), \quad (34)$$

with the non-negative integers  $r_1, r_2, t_1, t_2$  restricted according to  $0 \leq r_1 + r_2 \leq p - 3$ ,  $0 \leq t_1 + t_2 \leq q - 3$ .

<sup>1</sup> The level of a state is defined by  $\text{level} \equiv h + 3$ , where  $h$  is the weight of the fields in front of the exponential. For instance, the level of the state (29) is given by  $(-1 - 2) + 3 = 0$ . The state (29) is the state of lowest ghost number at level zero. Other level-zero states, with ghost number one higher, have a factor  $(\partial^2\gamma)(\partial\gamma)\gamma$  in front of the exponential.

For the above values of  $p_2, s_1$  and  $s_2$  we now calculate the conformal weight of the state (29). The conformal weight of a vertex operator  $\exp(ip\phi)$ , for a generic scalar  $\phi$  with background charge  $Q_\phi$ , is given by

$$h_\phi = \frac{1}{2}(p + iQ_\phi)^2 + \frac{1}{2}Q_\phi^2. \quad (35)$$

This, together with the weight  $-3$  of the ghost contribution to (29), determines the allowed values of the conformal weights at level 0.

Note that the weight due to the Liouville fields is invariant under the following transformation of the labels  $r_1, r_2, t_1, t_2$  [5]:

$$(r_1, r_2, t_1, t_2) \rightarrow (p - 3 - r_1 - r_2, r_1, q - 3 - t_1 - t_2, t_1). \quad (36)$$

Applying this transformation gives successively new Liouville momenta  $(s'_1, s'_2)$  and  $(s''_1, s''_2)$ , while applying (36) three times gives back the original momenta. A second transformation which leaves the Liouville weights invariant but changes the sign of the  $W_3$  weight is

$$(r_1, r_2, t_1, t_2) \rightarrow (r_2, r_1, t_2, t_1). \quad (37)$$

Note that  $(r_1, r_2, t_1, t_2)$  cannot be invariant under (36). Therefore the weights of states with momenta (33, 34) are either six-fold or three-fold degenerate, the possibility of three-fold degeneracy occurring when the  $W_3$  weight, which is proportional to [5]

$$\begin{aligned} & [q(r_1 - r_2) - p(t_1 - t_2)] [q(r_1 + 2r_2 + 3) - p(t_1 + 2t_2 + 3)] \\ & \times [q(2r_1 + r_2 + 3) - p(2t_1 + t_2 + 3)], \end{aligned} \quad (38)$$

vanishes. In a  $W_3$  minimal model, representations whose labels are related through (36) or (37) should be identified, since the corresponding irreducible  $W_3$  representations are isomorphic. In a  $W_3$  minimal model each of these should occur with multiplicity one only.

The three solutions (30)–(32) are in fact related by the transformation (36). We find

$$\begin{aligned} p_2(s'_1, s'_2)(A_0) &= p_2(s_1, s_2)(C_0), \\ p_2(s'_1, s'_2)(B_0) &= p_2(s_1, s_2)(A_0), \\ p_2(s'_1, s'_2)(C_0) &= p_2(s_1, s_2)(B_0). \end{aligned} \quad (39)$$

Therefore the weights obtained for the three solutions (30)–(32) are the same. In the next section we will show that these relations between (30)–(32) can be understood from the action of certain screening operators.

The conformal weights of the primary operators of the  $(p, q)$  Virasoro minimal model are given by

$$h_{\text{Vir}}(r, t) = \frac{(q(r+1) - p(t+1))^2 - (q-p)^2}{4pq}, \quad (40)$$

for  $0 \leq r \leq p-2$ ,  $0 \leq t \leq q-2$ .  $h_{\text{vir}}$  is invariant under the change of labels

$$r \rightarrow p-2-r, \quad t \rightarrow q-2-t. \quad (41)$$

Again, in a Virasoro minimal model, we should identify two representations whose labels are related through (41), since they give rise to isomorphic irreducible Virasoro modules.

We should now identify the calculated weights of (29) with (40). The following Virasoro weights are obtained:

$$\begin{aligned} (A_0) &\rightarrow h_{\text{vir}}(r_2, t_2), \\ (B_0) &\rightarrow h_{\text{vir}}(1+r_1+r_2, 1+t_1+t_2), \\ (C_0) &\rightarrow h_{\text{vir}}(r_1, t_1), \end{aligned} \quad (42)$$

with the values of  $r_1, r_2, t_1, t_2$  restricted by the conditions below (34). If we compare these restrictions with the Virasoro conditions below (40), we find that only the state  $h_{\text{vir}}(p-2, 0)$ , or, equivalently,  $h_{\text{vir}}(0, q-2)$ , is missing at level 0. The conformal weight of this missing state is  $(p-2)(q-2)/4$ .

The identification of the Virasoro minimal model by its primary operators therefore cannot be completed at level 0. For the case of the critical string and the Ising model, the authors of refs. [20–23] indeed found one of the three primary operators of the Ising model, of weight  $\frac{1}{2}$ , at level 1.

At level 1 we will consider operators of the form

$$V_1(p_2, s_1, s_2) = \gamma e^{ip_2\phi_2 + is_1\sigma_1 + is_2\sigma_2}. \quad (43)$$

The condition  $Q_1 V_1(p_2, s_1, s_2) = 0$  determines the values of the momenta for which  $V_1$  is a physical state<sup>1</sup>. This condition now comprises four quadratic equations in the momenta, of which the general solution is

$$\begin{aligned} (A_1) \quad p_2 &= -\frac{1}{2}(is_2 + iQ - \sqrt{6-Q^2}), \\ s_1 &= 0, \quad s_2 \text{ arbitrary}, \end{aligned} \quad (44)$$

$$\begin{aligned} (B_1) \quad p_2 &= is_2 - \frac{1}{2}(iQ + \sqrt{6-Q^2}), \\ s_1 &= \sqrt{3}s_2, \quad s_2 \text{ arbitrary}, \end{aligned} \quad (45)$$

$$\begin{aligned} (C_1) \quad p_2 &= is_2 - \frac{1}{2}(iQ + \sqrt{6-Q^2}), \\ s_1 &= -\sqrt{3}(s_2 + \frac{2}{3}i\sqrt{6-Q^2}), \quad s_2 \text{ arbitrary}. \end{aligned} \quad (46)$$

As we did for level 0, we now choose  $Q$  equal to (26), and restrict the momenta  $s_1, s_2$  to the allowed values (33), (34). However, at level 1 the momenta  $s_1$  and  $s_2$  are further restricted by (44)–(46). It is easy to see that these conditions, and the fact that  $p$  and  $q$  are relatively prime, imply

<sup>1</sup> The state  $V_1$  is the state of lowest ghost number at this level, and therefore cannot be written as  $Q_1$  acting on any other level 1 state.

$$s_1 = 0 \rightarrow r_2 = t_2 = 0, \quad (47)$$

$$s_1 = \sqrt{3}s_2 \rightarrow r_1 = t_1 = 0, \quad (48)$$

$$s_1 = -\sqrt{3}(s_2 + \frac{2}{3}i\sqrt{6-Q^2}) \rightarrow \text{no solution for } r_1, r_2, t_1, t_2. \quad (49)$$

Therefore only the solutions (44) and (45) remain. On calculating the weight of these states we find the following Virasoro weights:

$$(A_1) \rightarrow h_{\text{Vir}}(r_1, t_1 + 1), \quad (B_1) \rightarrow h_{\text{Vir}}(r_2 + 1, t_2). \quad (50)$$

The state  $h_{\text{Vir}}(p-2, 0)$  is present in  $(B_1)$ , the state  $h_{\text{Vir}}(0, q-2)$  in  $(A_1)$ . Therefore the state which was missing at level 0 is indeed found at level 1. Note that all the other Virasoro weights which were obtained at level 0 are again obtained at level 1, except the two states  $h_{\text{Vir}}(0, 0)$  and  $h_{\text{Vir}}(p-2, q-2)$  of weight 0.

The weights found in solutions  $(A_1)$  and  $(B_1)$  take on the same values. This can be understood from applying twice the transformation (36), or, more directly, from the invariance of (40) under the transformation  $r \rightarrow p-2-r, t \rightarrow q-2-t$ . The latter transformation gives a one-to-one map of the Virasoro weights corresponding to the  $(A_1)$  solution onto those corresponding to the  $(B_1)$  solution.

Thus all primary operators of the  $(p, q)$  Virasoro minimal model are obtained at level 0 and 1 of the  $\mathbf{Q}_1$  cohomology.

Before we start to discuss the role of picture-changing and screening operators in all of this, we will first try to understand the cohomology of  $\mathbf{Q}_1$  a bit further, in the spirit of ref. [3]. In this paper the BRST cohomology of the noncritical string is computed, using a Felder resolution of the minimal model in the matter sector, and using a general result for the cohomology of the BRST operator when acting on the tensor product of two Fock modules. The latter cohomology is quite simple, for generic momenta the cohomology contains only the tachyonic states, and for some special values of the momenta two extra states appear. These are related to the existence of singular vectors in the Fock modules. In the proof one replaces the matter and Liouville scalar fields  $\phi_M, \phi_L$  by linear combinations  $\phi_M \pm i\phi_L$ , which simplifies the analysis considerably.

Let us now present some evidence that a similar structure also exists in the case of the  $\mathbf{Q}_1$  cohomology acting on the tensor product of three Fock spaces. First, we are going to perform a transformation of the fields by defining

$$\chi_1 = \partial(\phi_2 - i\sigma_2), \quad (51)$$

$$\chi_2 = \partial(\phi_2 + \frac{1}{2}i\sigma_2 - \frac{1}{2}i\sigma_1\sqrt{3}), \quad (52)$$

$$\chi_3 = \partial(\phi_2 + \frac{1}{2}i\sigma_2 + \frac{1}{2}i\sigma_1\sqrt{3}). \quad (53)$$

The OPEs of the fields  $\chi_i$  read

$$\chi_i(z)\chi_j(w) \sim \frac{-\frac{3}{2}(1-\delta_{ij})}{(z-w)^2}. \quad (54)$$

In addition, we introduce the parameter  $\alpha$ , defined by

$$Q = \sqrt{\frac{3}{2}} \left( \alpha + \frac{1}{\alpha} \right), \quad (55)$$

$$\sqrt{6 - Q^2} = i\sqrt{\frac{3}{2}} \left( \alpha - \frac{1}{\alpha} \right). \quad (56)$$

For the  $(p, q)$  minimal  $W_3$  model, the parameter  $\alpha$  equals  $\sqrt{q/p}$ . In terms of these new variables and fields, the BRST current  $j_1$  reads

$$\begin{aligned} j_1 = \frac{2i}{3} \sqrt{\frac{2}{3}} \left\{ \gamma \left[ \chi_1 (\chi_2 \chi_3) - \frac{1}{\alpha} \sqrt{\frac{3}{2}} \partial (\chi_1 \chi_2) - \alpha \sqrt{\frac{3}{2}} \partial (\chi_2 \chi_3) \right. \right. \\ \left. \left. - \sqrt{\frac{3}{2}} \left( \alpha (\chi_1 \partial \chi_3) + \frac{1}{\alpha} (\partial \chi_1 \chi_3) \right) + \left( \frac{3}{2\alpha^2} - \frac{3}{4} \right) \partial^2 \chi_1 \right. \right. \\ \left. \left. + \frac{3}{4} \partial^2 \chi_2 + \left( \frac{3\alpha^2}{2} - \frac{3}{4} \right) \partial^2 \chi_3 \right] \right. \\ \left. + \frac{3}{2} \beta (\partial \gamma (\chi_1 + \chi_2 + \chi_3)) - \frac{3}{2} \sqrt{\frac{3}{2}} \left( \alpha + \frac{1}{\alpha} \right) \partial \beta (\partial \gamma \gamma) \right\}. \end{aligned} \quad (57)$$

To continue along the lines of ref. [3], we consider the mode expansion of  $Q_1$ , acting on a tensor product of three Fock modules and the ghost Hilbert space. We can decompose it in terms of ghost zero modes as

$$Q_1 = d - \beta_0 M + \gamma_0 W_0 + \gamma_0 \beta_0 H. \quad (58)$$

Compared to the ordinary noncritical string, an extra complication arises, in that a term containing  $\gamma_0 \beta_0$  appears. The condition that  $Q_1^2 = 0$  translates in the conditions

$$d^2 = M W_0, \quad (59)$$

$$M(d + H) = dM, \quad (60)$$

$$W_0 d = (d + H) W_0, \quad (61)$$

$$(d + H)^2 = W_0 M. \quad (62)$$

The operator  $d$  preserves the subspace of states annihilated by  $W_0$  and  $\beta_0$ , and on this space it satisfies  $d^2 = 0$ , so that one can consider the cohomology of  $d$  on this subspace (this cohomology is usually called the relative cohomology of  $Q_1$ ). In the case of the non-critical Virasoro string, there is a simple relation between the relative and absolute cohomology. Here that relation is less clear, since  $W_0$  is not diagonalizable in general. We will ignore this problem for the moment and focus only on  $d$ . It can be written as the sum of a piece containing

the terms linear in the oscillators and the ghosts and a set of remaining terms. We will be only interested in the piece linear in the ghosts and the oscillators. Denoting the modes of  $\chi_i$  by  $\eta_n^i$ , so that  $\chi_i = \sum \eta_n^i z^{-n-1}$ , we find for this part of  $d$

$$-\frac{2i}{3}\sqrt{\frac{2}{3}}\sum_{n\neq 0}\gamma_{-n}\left[\eta_n^1P_2^+(n)P_3(n)+\eta_n^2P_1(n)P_3(n)+\eta_n^3P_1(n)P_2^-(n)\right] \\ -\frac{i}{\sqrt{6}}\sum_{n\neq 0}\gamma_{-n}(n^2+n)(\eta_n^1+\eta_n^2+\eta_n^3), \quad (63)$$

where the polynomials  $P_1, P_2^\pm$  and  $P_3$  are given by

$$P_1(n) = p_2 - is_2 + i\sqrt{\frac{3}{2}}\alpha(n+2), \quad (64)$$

$$P_2^+(n) = p_2 + \frac{i}{2}s_2 - \frac{i}{2}s_1\sqrt{3} + i\sqrt{\frac{3}{2}}\left(\frac{n+1}{\alpha} + \alpha\right), \quad (65)$$

$$P_2^-(n) = p_2 + \frac{i}{2}s_2 - \frac{i}{2}s_1\sqrt{3} + i\sqrt{\frac{3}{2}}\left(\alpha(n+1) + \frac{1}{\alpha}\right), \quad (66)$$

$$P_3(n) = p_2 + \frac{i}{2}s_2 + \frac{i}{2}s_1\sqrt{3} + i\sqrt{\frac{3}{2}}\frac{n+2}{\alpha}. \quad (67)$$

The almost factorized form of  $d$  in (63) is very suggestive. Maybe one can define a modified operator  $d$  in such a way that the last term in (63) is absent. In any case numerical evidence shows that all the states in the cohomology of  $\mathbf{Q}_1$  have values of the momenta related to zeros of the polynomials  $P_i$ . Restricting attention to Virasoro primaries, we conjecture that the relative cohomology of  $\mathbf{Q}_1$  can be organized as follows:

- For generic momenta there is only one state at ghost number two and level zero, the standard tachyonic state, if  $P_1(0)P_2(0)P_3(0) = 0$ . (Here  $P_2(0) \equiv P_2^+(0) = P_2^-(0)$ .)
- If integers  $u_1, u_2 > 0$  exist such that (i)  $P_1(u_1) = P_2^+(u_2) = 0$ , (ii)  $P_2^-(u_1) = P_3(u_2) = 0$ , or (iii)  $P_1(u_1) = P_3(u_2) = 0$  there are two states at level  $u_1u_2$  and ghost numbers 2, 3.
- If integers  $u_1, u_2 < 0$  exist such that (i)  $P_1(u_1) = P_2^+(u_2) = 0$ , (ii)  $P_2^-(u_1) = P_3(u_2) = 0$ , or (iii)  $P_1(u_1) = P_3(u_2) = 0$  there are two states at level  $u_1u_2$  and ghost numbers 1, 2.
- If integers  $u_1, u_2, u_3, u_4$  exist such that  $P_1(u_1) = P_3(u_4) = 0$ , and  $P_2^-(u_2) + u_3\alpha = 0$ ,  $P_2^+(u_3) + u_2/\alpha = 0$ , there may be extra states at level  $u_1u_3 - u_3u_2 + u_2u_4$  and at three consecutive ghost numbers. The precise ghost numbers depend on the signs of the  $u_i$  but one of them is 2.



The three kinds of states are related to the six kinds of null states that minimal  $W_3$  representations generically have. These six are reduced to three here since we did not yet take the  $Q_0$  cohomology. This is all similar very to the analysis for the noncritical ordinary string, where a similar though simpler result holds for the cohomology of the BRST operator on the product of two Fock spaces and the ghost Hilbert space.

To prove this conjecture, and specify more precisely the cohomology for the last of the four cases, one could try to use the same spectral sequence techniques as in ref. [3]. In this respect it is suggestive that in terms of the  $\chi_i$  no term in  $Q_1$  contains any  $\chi_i$  more than once, and if one would assign a positive degree to say  $\chi_1$  and negative degree to the others, the first term in the spectral sequence would be the cohomology of the  $\sum \gamma_{-n} \eta_n^1$  term in (63). Apart from the problem with the extra term in (63), this would show that one needs zeros of the polynomials  $P_i$  to get extra states, but the details are not clear to us.

As an example, the third case with  $u_1 = u_2 = -1$  yields precisely the series of states (44)–(46).

#### 4. Screening and picture-changing operators

In the previous section we discussed in detail the  $Q_1$  cohomology for level 0 and 1, and for the particular ghost structures (29), (43). In addition, we conjectured for which momenta there will be extra states at other levels.

In the case of the critical  $W_3$  string it has been proposed that all these states can be obtained from the states at level 0 and 1 by acting on them with picture-changing and screening operators [20–23]. Generalizing the case of the critical string, we conjecture that with  $Q$  equal to  $Q_{\min}$  (26) and the momenta of the Liouville fields restricted to those of the  $(p, q)$  minimal model, all states in the  $Q_1$  cohomology will be related by picture changing and screening operators to the states obtained thus far, or to descendants of the primary operators discovered at level 0 and 1.

Screening operators (anti-)commute with  $Q_1$ , and have weight 0. They transform physical states into other physical states, provided the action of such an operator on a state is well-defined (see ref. [22], and the discussion below). Screening operators may change the level.

Picture changing operators are of the form  $[Q, \phi]$ , where  $\phi$  is a scalar field. They also produce solutions of the cohomology when acting on a physical state. In our case there are three such operators, corresponding to  $\phi_2, \sigma_1$  and  $\sigma_2$ . They have weight 0, change the ghost number of a physical state by a single unit but do not change the level. Applying the same picture-changing operator twice gives 0.

We will not prove the above conjecture in this paper. We will provide evidence for it by showing that the weights of the Virasoro  $(p, q)$  minimal model occur exactly once among the level 0 and 1 states which we obtained in the

previous section, if states related by screening and picture-changing operators are identified. This illustrates the use of these operators, and hopefully will stimulate efforts in obtaining a proof.

Let us first give a list of the screening operators which are relevant for our calculation. There are two screening operators of the form

$$S_i = \oint dz \beta(z) e^{i\alpha_{2,S_i}\phi_2(z) + i\beta_{1,S_i}\sigma_1(z) + i\beta_{2,S_i}\sigma_2(z)}. \quad (68)$$

These operators anticommute with  $Q_1$  provided the momenta  $\alpha_2, \beta_1, \beta_2$  take on the following values:

$$\begin{aligned} S_1 : \quad \alpha_{2,S_1} &= \frac{1}{3}\{iQ + \sqrt{6-Q^2}\}, \\ \beta_{1,S_1} &= \frac{1}{6}i\sqrt{3}\{iQ + \sqrt{6-Q^2}\}, \\ \beta_{2,S_1} &= \frac{1}{6}i\{iQ + \sqrt{6-Q^2}\}, \end{aligned} \quad (69)$$

$$\begin{aligned} S_2 : \quad \alpha_{2,S_2} &= \frac{1}{3}\{iQ - \sqrt{6-Q^2}\}, \\ \beta_{1,S_2} &= 0, \\ \beta_{2,S_2} &= -\frac{1}{3}i\{iQ - \sqrt{6-Q^2}\}. \end{aligned} \quad (70)$$

Then there are four screening operators which involve only the Liouville fields, and which are of the form

$$T_i = \oint dz e^{i\beta_{1,T_i}\sigma_1(z) + i\beta_{2,T_i}\sigma_2(z)}. \quad (71)$$

They commute with  $Q_1$  for the following values of the momenta:

$$\begin{aligned} T_1 : \quad \beta_{1,T_1} &= (Q - i\sqrt{6-Q^2})/\sqrt{3}, \\ \beta_{2,T_1} &= 0, \end{aligned} \quad (72)$$

$$\begin{aligned} T_2 : \quad \beta_{1,T_2} &= -(Q + i\sqrt{6-Q^2})/\sqrt{3}, \\ \beta_{2,T_2} &= 0, \end{aligned} \quad (73)$$

$$\begin{aligned} T_3 : \quad \beta_{1,T_3} &= \frac{1}{2}(-Q + i\sqrt{6-Q^2})/\sqrt{3}, \\ \beta_{2,T_3} &= -\frac{1}{2}(-Q + i\sqrt{6-Q^2}), \end{aligned} \quad (74)$$

$$\begin{aligned} T_4 : \quad \beta_{1,T_4} &= \frac{1}{2}(Q + i\sqrt{6-Q^2})/\sqrt{3}, \\ \beta_{2,T_4} &= -\frac{1}{2}(Q + i\sqrt{6-Q^2}). \end{aligned} \quad (75)$$

Finally, we have two screening operators of the form

$$R_i = \oint dz \gamma(z) e^{i\alpha_{2,R_i}\phi_2(z)}, \quad (76)$$

with the corresponding momenta:

$$R_1 : \alpha_{2,R_1} = -iQ - \sqrt{6 - Q^2}, \quad (77)$$

$$R_2 : \alpha_{2,R_2} = -iQ + \sqrt{6 - Q^2}. \quad (78)$$

The three picture-changing operators are  $P_{\phi_2} = [Q_1, \phi_2]$ ,  $P_{\sigma_1} = [Q_1, \sigma_1]$  and  $P_{\sigma_2} = [Q_1, \sigma_2]$ . We find for  $P_{\phi_2}$

$$P_{\phi_2} = -\frac{i}{3\sqrt{6}} \left[ 12\gamma(\partial\phi_2)^2 + 12Q(\partial\gamma)\partial\phi_2 + (-15 + 4Q^2)\partial^2\gamma \right] \\ + \frac{2i}{\sqrt{6}}\gamma T_L + i\sqrt{6}\gamma(\partial\gamma)\beta, \quad (79)$$

and similar expressions for  $P_{\sigma_1}$  and  $P_{\sigma_2}$ . In this section we consider only the action of these picture-changing operators on the vacuum state (29). The only term which then contributes is the  $\partial^2\gamma$  contribution, which is present in all three picture-changing operators. It produces physical states

$$\mathbf{V}_0(p_2, s_1, s_2) = (\partial^2\gamma)(\partial\gamma)\gamma e^{ip_2\phi_2 + is_1\sigma_1 + is_2\sigma_2}, \quad (80)$$

for the same values of the momenta obtained in (30)–(32). Since all picture-changing operators act the same way in this case, we will denote them collectively by  $P$ .

The action of the screening operators is more complicated. All our screening operators are of the form

$$S_i = \oint dz_i K_i(\beta(z_i), \gamma(z_i)) \exp \left[ i \sum_m p_{m,S_i} \phi_m(z_i) \right], \quad (81)$$

where  $\phi_m$  is a set of scalar fields, and  $p_{m,S_i}$  are the screening momenta in the operator  $S_i$  for the  $m$ th field. These operators act on states of the form

$$\mathbf{O} = L(\beta(w), \gamma(w)) \exp \left[ i \sum_m p_m \phi_m(w) \right]. \quad (82)$$

The condition under which this action is well-defined is discussed in detail in ref. [22]. If the action of the product of  $n$  such screening operators on  $\mathbf{O}$  is considered, then the number

$$P_n = n - 1 + \sum_m \sum_{i,j=1, i < j}^n p_{m,S_i} p_{m,S_j} + \sum_m \sum_{i=1}^n p_{m,S_i} p_m \quad (83)$$

should be an integer. This condition arises from the fact that the successive OPEs give, after appropriate changes in the integration variables  $z_i$ , rise to a single factor  $(z_1 - w)^{P_n}$ . The OPEs of the ghost contributions will similarly give a factor  $(z_1 - w)^{P_{\text{gh}}}$ , where  $P_{\text{gh}}$  is guaranteed to be an integer. The integral over  $z_1$  then gives a well-defined and non-trivial result if

$$P_n + P_{gh} = -1. \quad (84)$$

The momenta of the final state,  $\mathbf{O}' = S_1 \dots S_n \mathbf{O}$ , if it is defined, are equal to  $p_m + \sum_i p_{m,S_i}$ , and therefore the conformal weight of this proposed  $\mathbf{O}'$  can be calculated. Using the condition that the weight of the screening operators is zero, and, independently, that the weights of  $\mathbf{O}'$  and  $\mathbf{O}$  are equal, it is not difficult to show that

$$P_n = h_{L,\mathbf{O}} - h_{L,\mathbf{O}'} - 1 + \sum_i h_{K_i}, \quad (85)$$

where  $h_{K_i}$ ,  $h_{L,\mathbf{O}}$ ,  $h_{L,\mathbf{O}'}$  are the conformal weights of the ghost contributions to  $S_i$ ,  $\mathbf{O}$  and  $\mathbf{O}'$  respectively. Therefore, if the conformal weights of the initial and of the proposed final state are equal, and if the momenta of the screening operators interpolate between the momenta of initial and final states,  $P_n$  is automatically integer and can be easily calculated.

In discussing the action of the screening operators, it is useful to characterize their effect on the momenta, and on the labels  $r_1, r_2, t_1, t_2$  in the Liouville sector. Consider a screening operator

$$S = (R_1)^{l_1} (R_2)^{l_2} (S_1)^{m_1} (S_2)^{m_2} (T_1)^{n_1} (T_2)^{n_2} (T_3)^{n_3} (T_4)^{n_4}. \quad (86)$$

If we choose a  $(p, q)$   $W_3$  minimal model, so that  $Q = Q_{\min}$  with  $Q_{\min}$  given in (26), the changes in the momenta due to (86) are given by

$$\Delta p_2 = i [(2m_1 - 6l_1)q + (2m_2 - 6l_2)p] / \sqrt{6pq}, \quad (87)$$

$$\Delta s_1 = \sqrt{3} [(-m_1 + 2n_1 - n_3)q + (-2n_2 + n_4)p] / \sqrt{6pq}, \quad (88)$$

$$\Delta s_2 = [(-m_1 + 3n_3)q + (2m_2 - 3n_4)p] / \sqrt{6pq}. \quad (89)$$

The change in the Liouville momenta induces changes in the labels  $r_1, r_2, t_1, t_2$  in (33), (34). These are given by

$$\begin{aligned} \Delta r_1 &= n_1 - 2n_3 + lp, \\ \Delta r_2 &= m_1 - 2n_1 + n_3 + kp, \\ \Delta t_1 &= m_2 + n_2 - 2n_4 + lq, \\ \Delta t_2 &= -2n_2 + n_4 + kq, \end{aligned} \quad (90)$$

where  $l$  and  $k$  are integers chosen in such a way that the  $\Delta r_i$  and  $\Delta t_i$  produces labels in the allowed range  $0 \leq r_1 + r_2 \leq p - 3$ ,  $0 \leq t_1 + t_2 \leq q - 3$ . Of course, the action of  $S$  will be well-defined only if the initial and final conformal weights are equal.

Now that the action of the screening operators has been clarified, let us first discuss the way they act on the states at level 0. We choose a  $(p, q)$   $W_3$  minimal model. Then the Virasoro weights for, e.g., the physical states ( $A_0$ ), as given in (42), occur with a certain multiplicity. These weights are given by

$h_{\text{vir}}(r_2, t_2)$ , and thus independent of  $r_1$  and  $t_1$ . Using (90) it is easy to see that one can pass between the states of different  $r_1$  and  $t_1$  with screening operators. We have on the states ( $A_0$ ):

$$\begin{aligned} \text{On } (A_0) \quad S_2 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (0, 0, 1, 0), \\ S_1 T_1 T_3 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (-1, 0, 0, 0). \end{aligned} \quad (91)$$

The presence of the picture-changing operator  $P$  is required because the extra  $\partial^2 \gamma$  it introduces in  $V_0$  ensures that in the OPE with the screening operators (84) is satisfied.

In the previous section we showed that the physical states ( $A_0$ ), ( $B_0$ ) and ( $C_0$ ) are related by the discrete transformation (36). This induces transformations similar to (91) for the states ( $B_0$ ) and ( $C_0$ ). These can again be read off from (90) and are given by

$$\begin{aligned} \text{On } (B_0) \quad S_2 T_4 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (0, 0, -1, 1), \\ S_1 T_1 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (1, -1, 0, 0), \end{aligned} \quad (92)$$

$$\begin{aligned} \text{On } (C_0) \quad S_2 T_2 T_4 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (0, 0, 0, -1), \\ S_1 P \quad &\text{gives } \Delta(r_1, r_2, t_1, t_2) = (0, 1, 0, 0). \end{aligned} \quad (93)$$

In fact, the relation between (30)–(32) due to the discrete symmetry (36) can be represented by screening operators. Consider again the solutions ( $A_0$ ), with Virasoro weights  $h_{\text{vir}}(r_2, t_2)$ . If we perform a transformation (36) on the labels  $(r_i, t_i)$ , we find that  $1 + r'_1 + r'_2 = p - 2 - r_2$ ,  $1 + t'_1 + t'_2 = q - 2 - t_2$ . This means that  $h_{\text{vir}}(1 + r'_1 + r'_2, 1 + t'_1 + t'_2) = h_{\text{vir}}(p - 2 - r_2, q - 2 - t_2) = h_{\text{vir}}(r_2, t_2)$ . Therefore, with this transformation on the labels we find solutions among the ( $B_0$ ) states with the same weight as the ( $A_0$ ) state. Given this change in the labels we use (90) to obtain the corresponding screening operator. We find that from any physical state ( $A_0$ ) a state of type ( $B_0$ ) can be obtained using

$$\begin{aligned} V_{0, (B_0)}(p - 3 - r_1 - r_2, r_1, q - 3 - t_1 - t_2, t_1) \\ = (T_1)^{1+r_2} (T_2)^{1+t_2} (T_3)^{2+r_1+r_2} (T_4)^{2+t_1+t_2} V_{0, (A_0)}(r_1, r_2, t_1, t_2). \end{aligned} \quad (94)$$

In a similar way one obtains a relation between the states ( $C_0$ ) and ( $A_0$ ):

$$\begin{aligned} V_{0, (C_0)}(r_2, p - 3 - r_1 - r_2, t_2, q - 3 - t_1 - t_2) \\ = (T_1)^{2+r_1+r_2} (T_2)^{2+t_1+t_2} (T_3)^{1+r_1} (T_4)^{1+t_1} V_{0, (A_0)}(r_1, r_2, t_1, t_2). \end{aligned} \quad (95)$$

If we set up an equivalence relation between states, under which two states that are related by screening operators are equivalent, then at this stage we can limit ourselves at level 0 to the states ( $A_0$ ), with the further restriction that from states which differ only in the labels  $(r_1, t_1)$  only one representative is considered. The restrictions on  $(r_2, t_2)$  are then given by  $0 \leq r_2 \leq p - 3$ ,  $0 \leq t_2 \leq q - 3$ . The multiplicity of the remaining Virasoro weights can still be

equal to two. The weights with  $1 \leq r_2 \leq p-3$ ,  $1 \leq t_2 \leq q-3$  all occur twice, since both the labels  $(r_2, t_2)$  and  $(p-2-r, q-2-t)$  are in the allowed range. The labels with either  $r_2 = 0$  or  $t_2 = 0$  occur only once. This doubling occurs for  $p \geq 5$ ,  $q \geq 4$ .

This remaining doubling of the Virasoro weights can also be lifted by screening operators. We find for all  $(p, q)$ ,  $p \geq 5$ ,  $q \geq 4$

$$\begin{aligned} & \mathbf{V}_{0,(A_0)}(0, p-2-r_2, 0, q-2-r_2) \\ &= R_1(S_2)^2(T_1)^{r_2}(T_2)^{t_2+1}(T_3)^{p-2}(T_4)^q P \\ & \times \mathbf{V}_{0,(A_0)}(p-4-r_2, r_2, q-3-t_2, t_2). \end{aligned} \quad (96)$$

This relates one of the states with  $(r_2, t_2)$  to one of the states with  $(p-2-r_2, q-2-r_2)$ . Up to equivalence by screening operators, each Virasoro weight (except  $(p-2)(q-2)/4$ ) therefore occurs once and only once at level 0.

In this application of the screening operators at level 0 we choose the screening operators such that they give the required  $\Delta(r_1, r_2, t_1, t_2)$ . By construction, the screening operators interpolate between the initial and final momenta, and leave the weight invariant. Therefore, as explained above, the factor  $P_n$  (83) is an integer. In the application (91-93)  $P_n$  is equal to 2, while the OPE of the ghost  $\beta$  in  $S_i$  with  $(\partial^2\gamma)(\partial\gamma)\gamma$  gives  $P_{\text{gh}} = -3$  (as well as less singular terms). In the second application (94, 95) we find immediately  $P_n = -1$ . In the third application (96) we have an additional  $\gamma$  from  $R_1$ . Therefore  $P_n$  equals 3. The OPE of the ghosts gives  $P_{\text{gh}} = -4$ , leaving again the combination  $(\partial\gamma)\gamma$ . Therefore in all cases (84) is satisfied.

Now consider the states at level 1. Here we want to show that all states except those with momentum  $(p-2)(q-2)/4$  can be obtained from the level 0 states using screening operators. Since the momenta of all level 0 and level 1 states are given in the previous section, it is a simple matter to use again (87-90) to construct the appropriate screening operators. A useful hint about this choice follows from the values of the Virasoro weights. For instance, the states  $(A_1)$  have  $r_2 = t_2 = 0$ , and weight  $h_{\text{vir}}(r_1, t_1 + 1)$ . For the level 0 states with  $r_2, t_2 = 0$  the weights are  $h_{\text{vir}}(0, 0) = 0$ ,  $h_{\text{vir}}(1 + r_1, 1 + t_1)$ ,  $h_{\text{vir}}(r_1, t_1)$  for solutions  $(A_0)$ ,  $(B_0)$ ,  $(C_0)$ , respectively. Since  $h_{\text{vir}} = 0$  does not occur at level 1 (see the discussion at the end of Section 3), the states  $(A_1)$  cannot be obtained from states  $(A_0)$  with  $r_2 = t_2 = 0$ . They can be produced from  $(B_0)$  and/or  $(C_0)$ . For  $(B_0)$  we must choose  $\Delta r_1 = 1, \Delta t_1 = 0$ , for  $(C_0)$   $\Delta r_1 = 0, \Delta t_1 = -1$ , with of course  $\Delta r_2 = \Delta t_2 = 0$ .

This leads to the following result. The states  $(A_1)$  can be obtained from level 0 by the following operators:

$$\begin{aligned} \mathbf{V}_{1,(A_1)}(r_1, 0, t_1, 0) &= (S_2)^2 T_2 (T_4)^2 P \mathbf{V}_{0,(C_0)}(r_1, 0, t_1 + 1, 0) \\ &= (S_1)^2 T_1 P \mathbf{V}_{0,(B_0)}(r_1 - 1, 0, t_1, 0). \end{aligned} \quad (97)$$

Depending on the values of the labels on the right-hand side, both or only one of the above transitions is allowed. The picture-changing operator is again

required to ensure that (84) is satisfied. Note that the state  $V_{1,(A_1)}(0, 0, q-3, 0)$ , corresponding to the missing primary operator at level 0, cannot be obtained in this way because then the values of the labels on the right-hand side of (97) fall outside the allowed range.

Similarly, for the solutions  $B_1$  at level 1 we find that

$$\begin{aligned} V_{1,(B_1)}(0, r_2, 0, t_2) &= (S_1)^2 (T_1)^2 T_3 P V_{0,(A_0)}(0, r_2 + 1, 0, t_2) \\ &= (S_2)^2 T_4 P V_{0,(B_0)}(0, r_2, 0, t_2 - 1). \end{aligned} \quad (98)$$

As a last point, we must now show that a screening operator connects the two states with Virasoro weight  $(p-2)(q-2)/4$ , which does not occur at level 0. In fact, we can connect every state  $V_{1,(B_1)}$  to a state  $V_{1,(A_1)}$  by screening operators, analogously the relations (94-95) at level 0. A look at the form of the Virasoro weights (50) tells us that we should correlate the labels  $(r_1, 0, t_1, 0)$  for  $(A_1)$  and  $(0, r_2 = p-3-r_1, 0, t_2 = q-3-t_1)$  at  $(B_1)$ . This leaves the  $\phi_2$  momentum invariant, and corresponds to a transformation (36) in the Liouville sector. It can be easily seen that this correspondence is realized by:

$$\begin{aligned} V_{1,(B_1)}(0, p-3-r_1, 0, q-3-t_1) \\ = (T_1)^{2+r_1} (T_2)^{2+t_1} (T_3)^{1+r_1} (T_4)^{1+t_1} V_{1,(A_1)}(r_1, 0, t_1, 0). \end{aligned} \quad (99)$$

Thus all primary operators of the  $(p, q)$  Virasoro minimal model occur with multiplicity 1, up to screening and picture-changing operators.

## 5. Examples

In this section we will illustrate the results from the previous sections for the cases  $(p, q) = (4, 3)$  and  $(5, 4)$ .

In the first case  $c_3^{3;(4,3)} = 1/2$ , so that the Virasoro minimal model corresponds to the Ising model. The value of the background charge in this case is

$$(p, q) = (4, 3) \rightarrow Q = 21/\sqrt{72}, \quad \sqrt{6-Q^2} = -3i/\sqrt{72}. \quad (100)$$

This case has been much studied recently in the case of the critical  $W_3$  string [20–23]. In our construction, the  $W_3$  minimal model we start out with has  $c_L^{(4,3)} = 0$ , so that the Liouville sector corresponds to the “trivial”  $W_3$  minimal model.

In Table 2 we present the momenta and conformal weights for the physical states (30). As we saw in the previous section, the choices (31), (32) are connected to (30) by screening operators, and in this sense equivalent. The three possible choices of the labels  $r_1, r_2, t_1, t_2$  are related to each other by the discrete transformation (36), so that the Liouville weight  $h_L$  is equal for the three states. There is only a three-fold degeneracy, because under (37) no new

Table 2

Momenta and conformal weights for the states  $V_{0,(A_0)}$  (30) for  $p = 4$ ,  $q = 3$ . Note that all momenta in the table have been multiplied by a factor  $\sqrt{72}$ .  $h_L$  is the contribution of the Liouville sector to the total conformal weight  $h_{Vir}$ .

$(r_1, r_2, t_1, t_2)$	$p_2\sqrt{72}$	$s_1\sqrt{72}$	$s_2\sqrt{72}$	$h_L$	$h_{Vir}$
(0, 0, 0, 0)	$-18i$	0	0	0	0
(0, 1, 0, 0)	$-21i$	$-3\sqrt{3}$	-3	0	1/16
(1, 0, 0, 0)	$-24i$	0	-6	0	0

Table 3

Momenta and conformal weights for the states  $V_{1,(A_1)}$  (44) for  $p = 4$ ,  $q = 3$ .

$(r_1, r_2, t_1, t_2)$	$p_2\sqrt{72}$	$s_1\sqrt{72}$	$s_2\sqrt{72}$	$h_L$	$h_{Vir}$
(0, 0, 0, 0)	$-12i$	0	0	0	1/2
(1, 0, 0, 0)	$-9i$	0	-6	0	1/16

labels are generated. The Virasoro weight  $h_{Vir}$  for these states is completely determined by  $(r_2, t_2)$  (42), so that the equality of  $h_{Vir}$  for (0, 0, 0, 0) and (1, 0, 0, 0) is understood. In the previous section we showed that these two states can also be related by screening and picture-changing operators. Note that the  $\phi_2$  momenta for these two states are conjugate to each other, where conjugation of a momentum  $p_\phi$  for a field with background charge  $Q_\phi$  is defined as

$$(p_\phi)^* \equiv -p_\phi - 2iQ_\phi. \quad (101)$$

However, the Liouville momenta are not related by conjugation, since the lattice chosen in (33), (34) does not transform into itself under conjugation.

At level 0 we therefore find that the two available Virasoro weights occur with multiplicity 1, so that the operations (96) are not required.

The level 1 states of type  $V_{1,(A_1)}$  are given in Table 3. Here there are only two possible states. We showed in the previous section that the state (1, 0, 0, 0) is related to  $V_{0,(B_0)}$  (97), and, by implication, therefore also to  $V_{0,(A_0)}$ .

As we have seen, the (4, 3) case does not contain all the features discussed in the previous sections. The case  $(p, q) = (5, 4)$  corresponds more closely to the generic situation. In this case we have

$$(p, q) = (5, 4) \rightarrow Q = 27/\sqrt{120}, \quad \sqrt{6 - Q^2} = -3i/\sqrt{120}. \quad (102)$$

The Liouville sector corresponds to a  $W_3$  minimal model with  $c_L^{(5,4)} = 4/5$ , and the  $Q_1$  cohomology will result in the states of the  $c_3^{3;(5,4)} = 7/10$  Virasoro minimal model.

The number of physical states ( $A_0$ ) at level 0 is equal to

$$\binom{p-1}{2} \binom{q-1}{2} \quad (103)$$



Table 4

Momenta and conformal weights for the states  $V_{0,(A_0)}$  (30) for  $p = 5$ ,  $q = 4$ . Note that all momenta in the table have been multiplied by a factor  $\sqrt{120}$ .  $h_L$  is the contribution of the Liouville sector to the total conformal weight  $h_{\text{Vir}}$ .

$(r_1, r_2, t_1, t_2)$	$p_2\sqrt{120}$	$s_1\sqrt{120}$	$s_2\sqrt{120}$	$h_L$	$h_{\text{Vir}}$
(0, 0, 0, 0)	$-24i$	0	0	0	0
(0, 0, 0, 1)	$-19i$	$5\sqrt{3}$	5	$2/3$	$7/16$
(0, 0, 1, 0)	$-14i$	0	10	$2/3$	0
(0, 1, 0, 0)	$-28i$	$-4\sqrt{3}$	-4	$1/15$	$1/10$
(0, 1, 0, 1)	$-23i$	$\sqrt{3}$	1	$1/15$	$3/80$
(0, 1, 1, 0)	$-18i$	$-4\sqrt{3}$	6	$2/5$	$1/10$
(1, 0, 0, 0)	$-32i$	0	-8	$1/15$	0
(1, 0, 0, 1)	$-27i$	$5\sqrt{3}$	-3	$2/5$	$7/16$
(1, 0, 1, 0)	$-22i$	0	2	$1/15$	0
(0, 2, 0, 0)	$-32i$	$-8\sqrt{3}$	-8	$2/3$	$3/5$
(0, 2, 0, 1)	$-27i$	$-3\sqrt{3}$	-3	0	$3/80$
(0, 2, 1, 0)	$-22i$	$-8\sqrt{3}$	2	$2/3$	$3/5$
(1, 1, 0, 0)	$-36i$	$-4\sqrt{3}$	-12	$2/5$	$1/10$
(1, 1, 0, 1)	$-31i$	$\sqrt{3}$	-7	$1/15$	$3/80$
(1, 1, 1, 0)	$-26i$	$-4\sqrt{3}$	-2	$1/15$	$1/10$
(2, 0, 0, 0)	$-40i$	0	-16	$2/3$	0
(2, 0, 0, 1)	$-35i$	$5\sqrt{3}$	-11	$2/3$	$7/16$
(2, 0, 1, 0)	$-30i$	0	-6	0	0

Table 5

Momenta and conformal weights for the states  $V_{1,(A_1)}$  (44) for  $p = 5$ ,  $q = 4$ . Note that all momenta in the table have been multiplied by a factor  $\sqrt{120}$ .  $h_L$  is the contribution of the Liouville sector to the total conformal weight  $h_{\text{Vir}}$ .

$(r_1, r_2, t_1, t_2)$	$p_2\sqrt{120}$	$s_1\sqrt{120}$	$s_2\sqrt{120}$	$h_L$	$h_{\text{Vir}}$
(0, 0, 0, 0)	$-15i$	0	0	0	$7/16$
(0, 0, 1, 0)	$-20i$	0	10	$2/3$	$3/2$
(1, 0, 0, 0)	$-11i$	0	-8	$1/15$	$3/80$
(1, 0, 1, 0)	$-16i$	0	2	$1/15$	$3/5$
(2, 0, 0, 0)	$-7i$	0	-16	$2/3$	$3/80$
(2, 0, 1, 0)	$-12i$	0	-6	0	$1/10$

and therefore increases quadratically with  $p$  and  $q$ . The 18 states  $(A_0)$  for  $p = 5$ ,  $q = 4$  are presented in Table 4. The multiplicity of the Liouville weights is either 3 or 6, as explained in section 3.

The multiplicity of Virasoro weights  $h_{\text{Vir}}(r_2, t_2)$  is equal to

$$(p - 2 - r_2)(q - 2 - t_2) + r_2 t_2. \quad (104)$$

The second term is due to the possibility of making the transformation (41), and contributes only if  $r_2$  and  $t_2$  are both unequal to zero.

The screening operators which change  $r_1$  and  $t_1$  (91) lift part of the degeneracy. The only remaining double multiplicity occurs for the labels (0, 2, 0, 1)

and  $(0, 1, 0, 1)$  for weight  $3/80$ . This is lifted by the transformation (96).

Now consider the states  $(A_1)$  at level 1, with  $r_2 = t_2 = 0$ . The total number of states is therefore equal to  $(p-2)(q-2)$ . For any  $(p, q)$  the multiplicity of Virasoro weights is either 1 or 2, depending on the possibility of making the transformation (41). Multiplicity 2 occurs when  $r_1 \geq 1$  and  $t_1 \leq q-4$ . The Virasoro weight which was absent at level 0 occurs for  $r_1 = 0$ ,  $t_1 = q-3$ , and therefore has multiplicity 1. In Table 5 we give the states  $(A_1)$  at level 1. Note that indeed the Virasoro weight 0 is absent, and that the new weight  $3/2$  occurs with multiplicity 1. The only weight with multiplicity 2 is  $3/80$ . In the previous section we showed that all states except the one with Virasoro weight  $3/2$  can be obtained by screening operators from level 0.

## 6. Discussion

So far we have done all calculations in terms of free scalar fields with background charges, although we restricted the Liouville momenta to those corresponding to a  $W_3$  minimal model. Let us now discuss what happens if we compute the cohomology in case the Liouville sector is a  $W_3$  minimal model. To be able to apply the results obtained so far, we need a Felder resolution describing the irreducible  $W_3$  representations that constitute the minimal model. This resolution has been conjectured in ref. [34]. If we denote the Fock space of  $\sigma_1$  and  $\sigma_2$ , with momenta given by (33) and (34), by  $\mathcal{F}_{r_1, r_2; t_1, t_2}$ , then the Felder resolution is

$$\dots \mathcal{F}_{-2} \xrightarrow{Q_{-2}} \mathcal{F}_{-1} \xrightarrow{Q_{-1}} \mathcal{F}_0 \xrightarrow{Q_0} \mathcal{F}_1 \xrightarrow{Q_1} \dots \quad (105)$$

where  $Q_i$  is a sum of products of the screening operators  $T_i$ , and

$$\begin{aligned} \mathcal{F}_{2i} = & \bigoplus_{k_1+k_2+i=0} \mathcal{F}_{r_1+(2k_1-k_2)p, r_2+(2k_2-k-1)p; t_1, t_2} \\ & \oplus \bigoplus_{k_1+k_2+i=1} \mathcal{F}_{r_2+(2k_1-k_2)p, -4-r_1-r_2+(2k_2-k-1)p; t_1, t_2} \\ & \oplus \bigoplus_{k_1+k_2+i=1} \mathcal{F}_{-4-r_1-r_2+(2k_1-k_2)p, r_1+(2k_2-k-1)p; t_1, t_2} \end{aligned} \quad (106)$$

$$\begin{aligned} \mathcal{F}_{2i+1} = & \bigoplus_{k_1+k_2+i=0} \mathcal{F}_{-2-r_1+(2k_1-k_2)p, r_1+r_2+1+(2k_2-k-1)p; t_1, t_2} \\ & \oplus \bigoplus_{k_1+k_2+i=0} \mathcal{F}_{r_1+r_2+1+(2k_1-k_2)p, -2-r_2+(2k_2-k-1)p; t_1, t_2} \\ & \oplus \bigoplus_{k_1+k_2+i=1} \mathcal{F}_{-2-r_1+(2k_1-k_2)p, -2-r_2+(2k_2-k-1)p; t_1, t_2} \end{aligned} \quad (107)$$

Collectively, we denote this complex by  $\mathcal{F}_*(r_1, r_2; t_1, t_2)$ . The conjecture is that the zeroth cohomology of (105) is isomorphic to an irreducible  $W_3$  representation, and all other cohomologies vanish. We denote this symbolically by  $\mathcal{H}_{r_1, r_2; t_1, t_2}^{\min} = H_Q^*(\mathcal{F}_*(r_1, r_2; t_1, t_2))$ . An example, which is relevant to the critical  $W_3$  string, is to take the  $(4, 3)$   $W_3$  minimal model. In this case taking the cohomology with respect to the BRST operator in (105) should be the same as putting the fields  $\sigma_1 = \sigma_2 = 0$  by hand.

For a general minimal model, we are interested in the cohomology of  $Q_1$  acting on the tensor product of the ghost Hilbert space  $\mathcal{H}_{\beta, \gamma}$ , the Fock space  $\mathcal{H}_{p_2}^{\phi_2}$  and  $\mathcal{H}_{r_1, r_2; t_1, t_2}^{\min}$ . Using (105) this cohomology is

$$H_{Q_1}^*(\mathcal{H}_{\beta, \gamma} \otimes \mathcal{H}_{p_2}^{\phi_2} \otimes H_Q^*(\mathcal{F}_*(r_1, r_2; t_1, t_2))), \quad (108)$$

which, under certain assumptions, is the same as

$$H_Q^*(H_{Q_1}^*(\mathcal{H}_{\beta, \gamma} \otimes \mathcal{H}_{p_2}^{\phi_2} \otimes \mathcal{F}_*(r_1, r_2; t_1, t_2))). \quad (109)$$

This demonstrates that if the Liouville sector is a  $W_3$  minimal model, we have to drop the states in the  $Q_1$  cohomology that can be build out of other states using the screening operators  $T_i$  alone. However, states obtained by acting with the screening operators  $R_i$  and  $S_i$  represent new states in the noncritical  $W_3$  string. This is an important observation, as it shows that the noncritical  $W_3$  string does not simply reduce to an ordinary noncritical string theory. Actually, one starts to suspect that given any realization of a  $c = c_{\text{Vir}}^{(p, q)}$  Virasoro algebra, it is always possible to build a generalized Felder complex whose cohomology is precisely that of the associated Virasoro minimal model. In our case this generalized Felder complex contains the Fock spaces of  $\phi_2, \sigma_1, \sigma_2$  and the ghost Hilbert space  $\mathcal{H}_{\beta, \gamma}$ , and the generalized Felder BRST operator is composed of  $Q_1$  and the screening operators  $R_i, S_i, T_i$ . It would be interesting to see if one can make this conjecture more precise.

Once we know the  $Q_1$  cohomology, we can try to use this knowledge to compute the  $Q_0 + Q_1$  cohomology. Assuming the spectral sequence associated to the decomposition  $Q = Q_0 + Q_1$  collapses after the second term, the  $Q_0 + Q_1$  cohomology is the same as the  $Q_0$  cohomology of the  $Q_1$  cohomology. Now the  $Q_1$  cohomology is the direct sum of a set of Virasoro modules with respect to the stress-energy tensor  $T_L + T_{\phi_2} + T_{\beta, \gamma}$ . If we know what type of Virasoro modules these are, we can use the results for the noncritical string to find the  $Q_0$  cohomology on this space, since  $Q_0$  is simply the BRST operator of an ordinary noncritical string theory. Finally, given any state in the  $Q_0$  cohomology of the  $Q_1$  cohomology, we can use a standard tic-tac-toe construction [17] to construct representatives for the  $Q_0 + Q_1$  cohomology. It is straightforward to apply this to the states we computed in previous sections to obtain some of the known states in the  $Q_0 + Q_1$  cohomology.

Additional insight in the structure of the  $Q_1$  and  $Q_0$  cohomology can be obtained by the group-theoretical interpretation of the decomposition  $Q = Q_0 + Q_1$  sketched at the end of section 1. There we explained that the  $Q_1$  part

of  $\mathbf{Q}$  is related to a projection on the space perpendicular to the first simple root  $\alpha_1$ . To see what this implies for the cohomology, we notice that there is a natural ansatz [11] for the  $\mathbf{Q}$  cohomology, based on comparison with the ordinary noncritical string, see also refs. [10,12]. This ansatz reads as follows: represent the irreducible  $W_3$  representation labeled by  $(r_1, r_2; t_1, t_2)$  by two  $\mathfrak{sl}_3$  weights  $\Lambda_{r_1, r_2}^{(+)} = r_1 \Lambda_1 + r_2 \Lambda_2$  and  $\Lambda_{t_1, t_2}^{(-)} = t_1 \Lambda_1 + t_2 \Lambda_2$ . As usual,  $\alpha_i$  are the simple roots and  $\Lambda_i$  the fundamental weights of  $\mathfrak{sl}_3$ . Assume that we are dealing with the  $(p, q)$   $W_3$  minimal model, with  $\gcd(p, q) = 1$ , so that  $\alpha = \sqrt{q/p}$ . The matter momenta  $p_1, p_2$  can be combined to one weight

$$\Lambda_M = -i \left( \frac{p_1}{\sqrt{2}} \alpha_1 + p_2 \Lambda_2 \sqrt{\frac{3}{2}} \right). \quad (110)$$

Let  $W$  be the Weyl group,  $\hat{W}$  the affine Weyl group and  $\rho$  half the sum of the positive roots,  $\rho = \alpha_1 + \alpha_2$ . Then the ansatz reads that there is a quartet of states at ghost number  $3 + l_w(\hat{w})$ ,  $4 + l_w(\hat{w})$ ,  $4 + l_w(\hat{w})$  and  $5 + l_w(\hat{w})$ , if  $w \in W$  and  $\hat{w} \in \hat{W}$  exist such that

$$w^{-1} \left( \alpha \hat{w} * \Lambda^{(+)} - \frac{1}{\alpha} \Lambda^{(-)} + \left( \alpha - \frac{1}{\alpha} \right) \rho \right) = \left( \Lambda_M + \left( \alpha + \frac{1}{\alpha} \right) \rho \right). \quad (111)$$

Here,  $l_w(\hat{w})$  is the twisted length of  $\hat{w} \in \hat{W}$  [35]. If  $\hat{w} = t_\gamma w_0$ , with  $t_\gamma$  a translation in the  $\gamma$  direction,  $\gamma \in (\mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2)$ , and  $w_0 \in W$ , then

$$\hat{w} * \Lambda^{(+)} = w_0(\Lambda^{(+)} + \rho) - \rho + p\gamma. \quad (112)$$

The level of this quartet of states is given by

$$l = \left( \alpha \left[ \frac{\hat{w} * \Lambda^{(+)} + \Lambda^{(+)}}{2} \right] - \frac{\Lambda^{(-)}}{\alpha} + \left( \alpha - \frac{1}{\alpha} \right) \rho, \alpha \left( \hat{w} * \Lambda^{(+)} - \Lambda^{(+)} \right) \right). \quad (113)$$

As a non-trivial check, it has been verified [11] that this correctly reproduces all states in the critical  $W_3$  string [33], by putting  $\Lambda^{(+)} = \Lambda^{(-)} = 0$  in (111). The operators  $x$  and  $y$  of ref. [33] are related to particular translations  $t_\gamma$  in the root lattice. If we decompose (111) in a component in the  $\alpha_1$  direction and in the  $\Lambda_2$  direction, we get two equations. The  $\Lambda_2$  component determines  $p_2$ , and should be the equation that describes the  $\mathbf{Q}_1$  cohomology. The other equation describes the usual noncritical  $\mathbf{Q}_0$  Virasoro cohomology. A possible proof of (111) might consist of separately proving its  $\alpha_1$  and  $\Lambda_2$  components, using results for the  $\mathbf{Q}_1$  and  $\mathbf{Q}_0$  cohomology, although this obscures the group theoretical structure of (111). An important message is that by projecting (111) onto  $\alpha_1$ , which reduces the cohomology basically to that of the noncritical Virasoro string, we lose information. This information loss is accomplished by the screening operators  $R_i$  and  $S_i$ , if we use them to identify states.

Finally, let us summarize the main results and ideas put forward in this paper, generalized to an arbitrary  $W_N(p, q)$  minimal model coupled to  $W_N$  gravity. Denote by  $Q_N$  the BRST operator for this theory. Then

- There is a series of BRST operators  $Q_N^i$ ,  $i = 2, \dots, N$ , with  $Q_N^N \subset \dots \subset Q_N^2$ , such that  $(Q_N^n)^2 = 0$ , and  $Q_N^n$  involves only  $N - n + 1$  matter fields, all Liouville fields, and the ghosts system of spin  $n$  up to spin  $N$ .
- This decomposition induces a map  $H_{Q_N}^* \rightarrow H_{Q_N^n}^*$  which is neither surjective nor injective. In the root space of  $\mathfrak{sl}_N$ , this projection is given by a projection on the fundamental weights  $\Lambda_{n-1} \dots \Lambda_{N-1}$ .
- The cohomology  $H_{Q_N^n}^*$  forms a (reducible) module for the  $W_{n-1}$  algebra, and there is a map  $H_{Q_N^n}^* \rightarrow \mathcal{H}_{\min}^{p,q}(W_{n-1})$ , where the latter denotes the Hilbert space of the  $(p, q)$  minimal model for the  $W_{n-1}$  algebra. This map is surjective but not injective.
- There exists an additional set of screening operators acting on  $H_{Q_N^n}^*$ . If we identify the states in  $H_{Q_N^n}^*$  that are obtained by acting with these new screening operators, then the previous map turns into an isomorphism. This yields a new kind of resolution, different from the usual Felder one, for the  $(p, q)$  minimal models of the  $W_{n-1}$  algebra.
- The noncritical  $(p, q)$   $W_{n-1}$  string is a subsector of the  $(p, q)$  noncritical  $W_N$  string. A correlation function in the noncritical  $W_N$  string contains a correlation function of the noncritical  $W_n$  string, if one would use the extra screening operators previously mentioned to cancel the ghost number anomalies of the ghosts of spin  $n \dots N$ . In this way one avoids the problem of dealing with the  $W_N$  moduli. However, as we argued, it is not allowed to use these screening operators in the  $W_N$  string, and if the ghost number anomalies do not match correctly, one still needs an additional integration over the  $W_N$  moduli to compute these correlation functions. Thus, the  $W_{n-1}$  string does not solve the full  $W_N$  string.

A more detailed investigation of these statements will be left to future work.

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